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CFT on $\mathbb{P}^1$ and Monodromy Representations of the Braid Groups

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§0. From the differential equations of N-point functions of vertex operators in the conformal field theory on $\mathbb{P}^1$, arise the monodromy representations of the braid group B_N. In the meeting of last year, I reported that these monodromy representations give "all" irreducible representations of the Hecke algebra $H_N(q)$ of type A_{N-1} (obtained by H.Wenzl[W]) associated with the affine Lie algebra of type $A_n^{(1)}$. In this meeting, I will report that associated with the affine Lie algebras of type $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$, the monodromy representations of the group B_N give "all" irreducible representations of the Birman-Wenzl-Murakami algebra, the q-analogue of Brauer's centralizer algebras. Very important is Jimbo-Miwa-Okado's calculations[JMO], and in the case of type $C_n^{(1)}$ the representations are equivalent to the ones obtained by J.Murakami[M].

§1. Let $\mathfrak{g}$ be the simple Lie algebra of type X_n, and $\hat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c$ be the affine Lie algebra of type $X_n^{(1)}$. Fix an integer $\ell\geq 1$ and introduce the number $\kappa=\ell+g$, where g is the dual Coxeter number of $\hat{\mathfrak{g}}$.

Denote by P_+ the set of dominant integral weights of $\mathfrak{g}$ and by P_ℓ the set of elements $\lambda\in P_+$ satisfying $(\theta,\lambda)\leq\ell$, where θ is the maximum root. For a weight $\lambda \in P_\ell$, we denote by V_λ the irreducible representation of $\mathfrak{g}$ of highest weight λ, by $\mathscr{H}_\lambda$ the integrable representation of $\hat{\mathfrak{g}}$ of highest weight $\ell\Lambda_0+\lambda$ and by $|\lambda\rangle$ the (fixed) highest weight cyclic vector of V_λ and $\mathscr{H}_\lambda$.

The Virasoro algebra also acts on $\mathscr{H}_\lambda$ by the Sugawara forms $L(m), m\in\mathbb{Z}$, and the space $\mathscr{H}_\lambda$ is graded by means of the eigenspace decomposition w.r.t. the operator $L(0)$:

$$\mathscr{H}_\lambda = \sum_{d\in\mathbb{Z}_{\geq 0}} \mathscr{H}_\lambda(d) \ , \ \mathscr{H}_\lambda(d) = \{v\in\mathscr{H}_\lambda ;\ L(0)v = (\Delta_\lambda+d)v\} \ ,$$

where $\Delta_\lambda = \dfrac{(\lambda,\lambda+2\rho)}{2\kappa}$ and ρ is the half sum of positive roots of $\mathfrak{g}$. Note that $\dim\mathscr{H}_\lambda(d)<\infty$ and $\mathscr{H}_\lambda(0) \cong V_\lambda$.

There are dual right $\mathfrak{g}$ and $\hat{\mathfrak{g}}$-module V_λ^* and $\mathcal{H}_\lambda^*$ of V_λ and $\mathcal{H}_\lambda$, and the nondegenerate invariant bilinear form $\langle\ |\ \rangle$ on $V_\lambda^* \times V_\lambda$ and $\mathcal{H}_\lambda^* \times \mathcal{H}_\lambda$ with the normalized condition $\langle\lambda|\lambda\rangle = 1$ where $\langle\lambda|$ is a fixed highest weight vector of $V_\lambda^* = \mathcal{H}_\lambda^*(0)$ and $\mathcal{H}_\lambda^*$.

A triple $\mathbb{V} = \begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{pmatrix}$ of weights in P_ℓ is called a vertex and is drawn as

$$\mathbb{V} = \begin{tikzcd} & \lambda \arrow[d] & \\ \mu_2 & \cdot \arrow[l] & \mu_1 \arrow[l] \end{tikzcd} .$$

A multi-valued, holomorphic function

$$\Phi(z) : V_\lambda \otimes \mathcal{H}_{\mu_1} \longrightarrow \hat{\mathcal{H}}_{\mu_2} = \prod_{d \in \mathbb{Z}_{\geq 0}} \mathcal{H}_{\mu_2}(d)$$

on $\mathbb{P}^1 \setminus \{0, \infty\}$ is called a *vertex operator of type* $\begin{pmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{pmatrix}$ (sometimes called *of weight* λ), if it satisfies the following:

(Gauge Condition) $[X(m),\ \Phi(z)(u\otimes\cdot)] = z^m \Phi(z)(Xu\otimes\cdot)$

$(X \in \mathfrak{g}, m \in \mathbb{Z}, u \in V_\lambda)$;

(Eq.of Motion) $[L(m),\ \Phi(z)] = z^m\{z\frac{d}{dz} + (m+1)\Delta_\lambda\}\Phi(z)$, where $X(m)=X\otimes t^m$ and the number Δ_λ is called the *conformal dimension* of the vertex operator $\Phi(z)$.

Denote by $\mathcal{V}er(\mathbb{v})$ the space of all vertex operators of type $\mathbb{v}$, and introduce the space

$$\mathcal{V}er(\lambda) = \sum_{\mu_1,\mu_2\in P_\ell} \mathcal{V}er\left(\begin{bmatrix}\lambda \\ \mu_2,\mu_1\end{bmatrix}\right)$$

of all vertex operators of weight λ.

Introduce the subalgebra $\mathfrak{k}_\theta = \mathbb{C}\langle X_\theta, [X_\theta, X_{-\theta}], X_{-\theta}\rangle \cong \mathfrak{sl}(2;\mathbb{C})$ of $\mathfrak{g}$ and the subspace $\mathcal{V}(\mathbb{v})$ of $\mathrm{Hom}_{\mathfrak{g}}(V_\lambda\otimes V_{\mu_1}, V_{\mu_2})$ defined by

$$\mathcal{V}(\mathbb{v}) = \cap \operatorname{Ker} \pi_{\mathfrak{k}_\theta}(j,j_1,j_2)$$

where the intersection is taken over the set $\{j,j_1,j_2\in \frac{1}{2}\mathbb{Z}_{\geq 0};\ j+j_1+j_3>\ell\}$, and $\pi_{\mathfrak{k}_\theta}(j,j_1,j_2)(\varphi) \in \mathrm{Hom}_{\mathfrak{k}_\theta}(W_j\otimes W_{j_1}, W_{j_2})$ is defined as

$$\pi_{\mathfrak{k}_\theta}(j,j_1,j_2)(\varphi) = \mathrm{proj}_{W_{j_2}}\circ\varphi\big|_{W_j\otimes W_{j_1}} \qquad (\varphi\in\mathrm{Hom}_{\mathfrak{g}}(V_\lambda\otimes V_{\mu_1}, V_{\mu_2}))$$

where W_j, W_{j_1}, W_{j_2} are $\mathfrak{k}_\theta$-simple submodules of $V_\lambda, V_{\mu_1}, V_{\mu_2}$ with spin j,j_1,j_2 respectively.

By Equation of Motion, Φ is expressed as a formal

Laurent series

$$\Phi(z) = \sum_{m\in\mathbb{Z}} \Phi(m)\ z^{-m-\hat{\Delta}(\mathbb{v})} ,$$

where $\hat{\Delta}(\mathbb{v}) = \Delta_\lambda+\Delta_{\mu_1}-\Delta_{\mu_2}$ and $\Phi(m)$ is homogeneous of degree m, i.e.

$$\Phi(m):\ V_\lambda\otimes\mathcal{H}_{\mu_1}(d) \longrightarrow \mathcal{H}_{\mu_2}(d-m) \qquad \text{for any d.}$$

The principal branch of $\Phi(z)$ is taken such as the value of $z^{-\hat{\Delta}(\mathbb{v})}$ is positive for $z\in\mathbb{R}_+=\{z\in\mathbb{R};\ z>0\}$ and uniquely continued to the region $\mathbb{C}_+=\{z\in\mathbb{C};\ \mathrm{Im}z>0\}$, and we refer this for the value of $\Phi(z)$ on $\mathbb{C}_+$.

For any vertex operator $\Phi\in\mathcal{V}e\imath(\mathbb{v})$, its initial term φ = $\Phi(0)|_{V_\lambda\otimes\mathcal{H}_{\mu_1}(0)} = \mathrm{proj}_{V_{\mu_2}}\circ z^{\hat{\Delta}(\mathbb{v})}\Phi(z)\ |_{V_\lambda\otimes V_{\mu_1}}$ belongs to $\mathcal{V}(\mathbb{v})$. Under this correspondence,

Theorem 1. *The space* $\mathcal{V}e\imath(\mathbb{v})$ *of N-point functions of type* $\mathbb{v}$ *is isomorphic with the space* $\mathcal{V}(\mathbb{v})$ *of initial terms of type* $\mathbb{v}$.

Call $\mathbb{v}$ ℓCG *(ℓ-constrained Clebsch-Gordan)* vertex, if $\mathcal{V}(\mathbb{v}) \neq 0$, and denote by ($\ell$CG) the set of all ℓCG vertices.

For each $\varphi \in \mathcal{V}(\mathbf{v})$, denote by Φ_{φ} the vertex operator with the initial term φ.

Notes. i) Even if we asuume that $\lambda \in P_+$ and $\mu_i \in P_\ell$, $\mathcal{V}\left(\begin{bmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{bmatrix}\right) \neq 0$ implies that $\lambda \in P_\ell$.

ii) Operator product expansions of *currents* $X(z) = \sum_{m \in \mathbb{Z}} X(m) z^{-m-1}$ $(X \in \mathfrak{g})$ and the *energy-momentum tensor* $T(z) = \sum_{m \in \mathbb{Z}} L(m) z^{-m-2}$ with vertex operators allow the extension of the vertex operators $\Phi(z)$ of type $\begin{bmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{bmatrix}$ to the operators $\Phi(z)$: $\mathcal{H}_\lambda \otimes \mathcal{H}_{\mu_1} \longrightarrow \hat{\mathcal{H}}_{\mu_2}$ by means of contour integrals. (*Nuclear Democracy*)

iii) By the same arguments as in §3, the analytic continuation of a vertex operator Φ of type $\begin{bmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{bmatrix}$ along the path γ_0 gives a vertex operator of type $\begin{bmatrix} & \lambda & \\ \mu_1 & & \mu_2 \end{bmatrix}$, where

$$\gamma_0(t) = z\, e^{\pi\sqrt{-1}t} \quad , \quad t \in [0,1] \ , \ z \in \mathbb{R}_+ \ .$$

This gives an isomorphism C_{γ_0} of $\mathcal{V}er\begin{bmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{bmatrix}$ to $\mathcal{V}er\begin{bmatrix} & \lambda & \\ \mu_1 & & \mu_2 \end{bmatrix}$ and the corresponding isomorphism

$$C_{\gamma_0} : \mathcal{V}\begin{bmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{bmatrix} \longrightarrow \mathcal{V}\begin{bmatrix} & \lambda & \\ \mu_1 & & \mu_2 \end{bmatrix}$$

is given by

$$C_{\gamma_0} = e^{\pi\sqrt{-1}\ \hat{\Delta}(\mathbb{v})}\,T\ ,$$

where T is the transposition:

$$T : \mathrm{Hom}(V_\lambda \otimes V_{\mu_1}, V_{\mu_2}) \longrightarrow \mathrm{Hom}(V_{\mu_1} \otimes V_\lambda, V_{\mu_2})$$

$$(T\varphi)(u\otimes v) = \varphi(v\otimes u).$$

A vertex operator $\Phi(z)$ of type $\mathbb{v}$ is also considered as an operator from $\mathcal{H}_{\mu_1}$ to $\hat{\mathcal{H}}_{\mu_2}$ parametrized by V_λ, i.e.

$$\Phi(u;z)(v) = \Phi(z)(u\otimes v) \qquad (u\in V_\lambda,\ v\in \mathcal{H}_{\mu_1}).$$

§2. It is convenient to introduce the spaces $\mathcal{H} = \sum_{\lambda\in P_\ell} \mathcal{H}_\lambda$ and $\hat{\mathcal{H}} = \sum_{\lambda\in P_\ell} \hat{\mathcal{H}}_\lambda$ and consider vertex operators as linear operators of $\mathcal{H}$ to $\hat{\mathcal{H}}$. The vacuum $|0\rangle$ of $\mathcal{H}_0$ is called a *Virasoro vacuum*, since $L(m)|0\rangle = 0$ for $m\geq -1$. Note that $V_0 = \mathbb{C}|0\rangle$.

For an N-ple $\mathbf{\Lambda}=(\lambda_N,\cdots,\lambda_1)$ of weights in P_ℓ, denote

$$V_{\mathbf{\Lambda}} = V_{\lambda_N}\otimes\cdots\otimes V_{\lambda_1} \quad \text{and} \quad V_{\mathfrak{g}}^{\vee}(\mathbf{\Lambda}) = \mathrm{Hom}_{\mathfrak{g}}(V_{\mathbf{\Lambda}},\mathbb{C})\ .$$

For any vertex operators $\Phi^i(z_i)$ of weight $\lambda_i (1\leq i\leq N)$,

$$\langle 0|\Phi^N(z_N)\cdot\cdots\cdot\Phi^1(z_1)|0\rangle$$

is the coefficient of $|0\rangle$ the iterated application $\Phi^N(z_N)\cdot\cdots\cdot\Phi^1(z_1)|0\rangle$ to the vector $|0\rangle$, and this is a $V_g^\vee(\Lambda)$-valued formal Laurent series in $z_N,\cdots,z_1$ called the *N-point function of weight* Λ and is denoted by $\langle\Phi^N(z_N)\cdots\Phi^1(z_1)\rangle$. Denote by $\mathcal{Ver}(\Lambda)$ the space of all N-point functions of weight Λ.

The space $V_g^\vee(\Lambda)$ is decomposed as

$$V_g^\vee(\Lambda) = \sum_{\mu} V_g^\vee(\Lambda)_\mu , \qquad \mu=(\mu_{N-1},\cdots,\mu_1) \in (P_+)^{N-1} ;$$

$$V_g^\vee(\Lambda)_\mu \xleftarrow[\cong]{C_\Lambda} \mathrm{Hom}_g(V_{\lambda_N}\otimes V_{\mu_{N-1}},V_0)\otimes\cdots\otimes\mathrm{Hom}_g(V_{\lambda_i}\otimes V_{\mu_{i-1}},V_{\mu_i})$$
$$\otimes\cdots\otimes\mathrm{Hom}_g(V_{\lambda_1}\otimes V_0,V_{\mu_1}),$$

where the identification C_Λ is given by

$$\begin{aligned} C_\Lambda(\varphi_N\otimes\cdots\otimes\varphi_1)(u_N\otimes\cdots\otimes u_1) &= \langle 0|\varphi_N(u_N\otimes\varphi_{N-1}(\cdots\otimes\varphi_2(u_2\otimes\varphi_1(u_1\otimes|0\rangle)\cdot\cdot)\rangle \\ &= \langle 0|\varphi_N(u_N)\circ\cdots\circ\varphi_1(u_1)(|0\rangle) ,\end{aligned}$$

for $\varphi_i\in \mathrm{Hom}_g(V_{\lambda_i}\otimes V_{\mu_{i-1}},V_{\mu_i}) \cong \mathrm{Hom}_g(V_{\lambda_i},\mathrm{Hom}(V_{\mu_{i-1}},V_{\mu_i}))$ $(1\le i\le N;\ \mu_N=\mu_0=0)$, and $u_N\otimes\cdots\otimes u_1\in V_\Lambda$.

Introduce the subspace $\mathcal{V}(\Lambda)$ of $V_{\mathfrak{g}}^{\vee}(\Lambda)$ defined, through C_{Λ}, by

$$\mathcal{V}(\Lambda) = \sum_{\mu} \mathcal{V}(\Lambda)_{\mu}, \qquad \mu=(\mu_{N-1},\cdots,\mu_1) \in (P_{\ell})^{N-1} ;$$

where

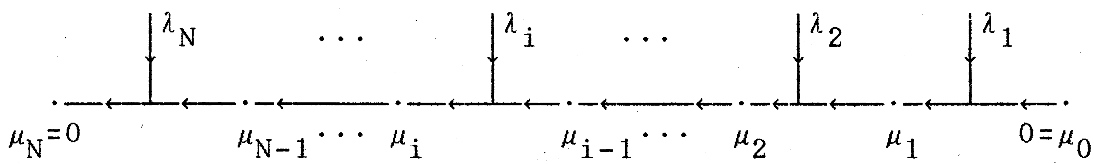
$$\mathcal{V}(\Lambda)_{\mu} = \mathcal{V}(v_N(\mu))\otimes\cdots\otimes\mathcal{V}(v_i(\mu)) \otimes\cdots\otimes\mathcal{V}(v_1(\mu)) \subset V_{\mathfrak{g}}^{\vee}(\Lambda)$$

and

$$v_N(\mu)=\begin{bmatrix} & \lambda_N & \\ 0 & & \mu_{N-1}\end{bmatrix},\cdots,\ v_i(\mu)=\begin{bmatrix} & \lambda_i & \\ \mu_i & & \mu_{i-1}\end{bmatrix},\cdots,\ v_1(\mu)=\begin{bmatrix} & \lambda_1 & \\ \mu_1 & & 0\end{bmatrix}.$$

Then the space $\mathcal{V}(\Lambda)$ is isomorphic to $\mathcal{V}e\imath(\Lambda)$ of N-point functions of weight Λ as follows: to each $\varphi = C(\varphi_N\otimes\cdots\otimes\varphi_1)\in\mathcal{V}(\Lambda)$, assign the N-point function

$$\Phi_{\varphi_N\otimes\cdots\otimes\varphi_1}(z) = \langle\Phi_{\varphi_N}(z_N)\cdots\Phi_{\varphi_1}(z_1)\rangle \in \mathcal{V}e\imath(\Lambda) .$$



Now introduce a system $KZ(\Lambda)$ of differential equations on $\mathrm{Hom}_{\mathfrak{g}}(V_{\Lambda},\mathbb{C})$-valued functions $\Phi(z)$ on $X_N = \{z=(z_N,\cdots,z_1) \in\mathbb{C}^N;\ z_i\neq z_j\ (i\neq j)\}$

$$\mathrm{KZ}(\Lambda) \qquad \left(\kappa\frac{\partial}{\partial z_i} - \sum_{\substack{k=1\\k\neq i}}^{N} \frac{\Omega_{ik}}{z_i - z_k}\right) \Phi(z) = 0 \qquad (1\le i\le N)$$

due to Knizhnik-Zamolodchikov[KZ], where

$$\Omega_{ik} = \sum_{a=1}^{\dim\mathfrak{g}} \rho_i(X^a)\ \rho_k(X_a)\ ,$$

ρ_i denotes the $\mathfrak{g}$-action on the i-th component of $\mathrm{Hom}(V_\Lambda, \mathbb{C})$ and $\{X^a\}$ and $\{X_a\}$ are dual bases of $\mathfrak{g}$.

Further introduce an additional ℓ-constraint condition, *i.e.* a system $\ell C(\Lambda)$ of algebraic equations

$$\ell C(\Lambda) \quad \sum_{|\mathbb{m}_i|=L_i} \binom{L_i}{\mathbb{m}_i} \prod_{k\neq i} (z_k - z_i)^{-m_k} \Phi(z)(X_\theta^{m_N} u_N, \cdot\cdot, |\lambda_i\rangle, \cdot\cdot, X_\theta^{m_1} u_1) = 0,$$

$$(1\le i\le N)$$

for any $u_k \in V_{\lambda_k}$ $(k\neq i)$, where $\mathbb{m}_i = (m_N, \cdots, \hat{m}_i, \cdot\cdot, m_1) \in (\mathbb{Z}_{\ge 0})^{N-1}$, $|\mathbb{m}_i| = \sum_{k\neq i} m_k$, $L_i = \ell - (\lambda_i, \theta) + 1$ and $\binom{L_i}{\mathbb{m}_i}$ is the multinomial coefficient.

Remark. The system $\mathrm{KZ}(\Lambda)$ of differential equations is completely integrable because of the infinitesimal pure braid relations among the operators Ω_{ik} (see [A]). The system $\ell C(\Lambda)$ is compatible with the system $\mathrm{KZ}(\Lambda)$.

Any N-point function of weight Λ satisfies the systems of $KZ(\mathbb{\Lambda})$ and $\ell C(\mathbb{\Lambda})$. Hence

Theorem 2.

i) For any N-ple $\mathbb{\Lambda}$ *of weights in* P_ℓ, *any N-point function* $\langle \Phi^N(z_N)\cdots\Phi^1(z_1)\rangle$ of weight $\mathbb{\Lambda}$ *is absolutely convergent in the region* $\mathcal{R}_N$, *and is analytically continued to a multivalued holomorphic function on* X_N, *where* $\mathcal{R}_N$ *is defined by*

$$\mathcal{R}_N = \{\mathbb{z}=(z_N,\cdots,z_1)\in\mathbb{C}_+^N;\ |z_N|>\cdots>|z_1|\} \subset X_N .$$

ii) The solution space of the joint system $KZ(\mathbb{\Lambda})$ *and* $\ell C(\mathbb{\Lambda})$ *is isomorphic with* $\mathcal{V}_{e\lambda}(\mathbb{\Lambda})$, *hence with* $\mathcal{V}(\mathbb{\Lambda})$.

Note. If $\mathbb{v} = \begin{bmatrix} & \lambda & \\ \mu & & 0\end{bmatrix} \in (CG)$, then $\mu = \lambda$, $\hat{\Delta}(\mathbb{v}) = 0$, and $\mathcal{V}(\mathbb{v}) \cong \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_\lambda) = \mathbb{C}\mathrm{id}$.

If $\mathbb{v} = \begin{bmatrix} & \lambda & \\ 0 & & \mu\end{bmatrix} \in (CG)$, then $\mu = \lambda^+$, $\hat{\Delta}(\mathbb{v}) = 2\Delta_\lambda$, and $\mathcal{V}(\mathbb{v}) \cong \mathrm{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\lambda^+}, \mathbb{C}) = \mathbb{C}\nu$, where the *anti-weight* λ^+ of λ is defined as $-\lambda^+(=w_0\lambda)$ is the lowest weight of V_λ and ν is normalised as $\nu(|\lambda\rangle\otimes w_0|\lambda^+\rangle)=1$, where w_0 is the longest element of the Weyl group of $\mathfrak{g}$.

3-point functions are essentially nothing but vertex operators. The assignation to $\varphi \in \mathcal{V}\left(\begin{bmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{bmatrix}\right)$ the element

$\nu\otimes\varphi\otimes\mathrm{id} \in \mathcal{V}(\mu_2^{+}, \lambda, \mu_1)$,

$$\nu\otimes\varphi\otimes\mathrm{id}(|u\rangle\otimes|v\rangle\otimes|w\rangle)$$
$$=\nu(|u\rangle\otimes\varphi(|v\rangle\otimes|w\rangle)) \qquad \left(|u\rangle\otimes|v\rangle\otimes|w\rangle\in V_{\mu_2^{+}}\otimes V_{\lambda}\otimes V_{\mu_1}\right),$$

gives the isomorphism between them. Hence the space $\mathcal{Ver}\left(\begin{bmatrix} & \lambda & \\ \mu_2 & & \mu_1 \end{bmatrix}\right)$ of vertex operators is isomorphic with the space $\mathcal{Ver}(\mu_2^{+}, \lambda, \mu_1)$ of 3-point functions. More precisely, the classical sector $\mathrm{proj}_{V_{\mu_2}} \circ \Phi_{\varphi}(z)|_{V_\lambda\otimes V_{\mu_1}}$ of the vertex operator $\Phi_{\varphi}(z)$ is given by

$$\lim_{z_t\nearrow\infty}\lim_{z_s\searrow 0} z^{2\Delta_{\mu_2}} \langle\Phi_{\nu}(z_t)\Phi_{\varphi}(z)\Phi_{\mathrm{id}}(z_s)\rangle.$$

§3. Denote by $\mathcal{Ver}(\mathbb{v}_2)\circ\mathcal{Ver}(\mathbb{v}_1)$ the space of compositions $\Phi^2(z_2)\Phi^1(z_1)$ of vertex operators Φ^i of type $\mathbb{v}_i$. Then

$$\sum_{\mu\in P_\ell} \mathcal{Ver}\left(\begin{bmatrix} & \lambda_2 & \\ \mu_t & & \mu \end{bmatrix}\right)\circ\mathcal{Ver}\left(\begin{bmatrix} & \lambda_1 & \\ \mu & & \mu_s \end{bmatrix}\right) \cong \mathcal{Ver}(\Lambda) \cong \mathcal{V}(\Lambda) \ ,$$

where $\Lambda = (\mu_t^{+}, \lambda_2, \lambda_1, \mu_s)$.

The composition $\Phi^2(z_2)\Phi^1(z_1)$ is determined by the classical sector $\mathrm{proj}_{V_{\mu_t}}\circ\Phi^2(z_2)\Phi^1(z_1)|_{V_{\mu_s}} \in$ $\mathrm{Hom}_{g}(V_{\lambda_2}\otimes V_{\lambda_1}\otimes V_{\mu_s},\ V_{\mu_t})$ and it is given by

$$\lim_{z_t \nearrow \infty} \lim_{z_s \searrow 0} z_t^{2\Delta_{\mu_t}} \langle \Phi_\nu(z_t)\Phi^2(z_2)\Phi^1(z_1)\Phi_{id}(z_s)\rangle .$$

Hence by Theorem 2, the composition $\Phi^2(z_2)\Phi^1(z_1)$ is absolutely convergent in the range $\mathscr{R}_2 = \{(z_2,z_1)\in\mathbb{C}_+^2;\ |z_2|>|z_1|>0\}$, so by the analytic continuation it defines the holomorphic (multivalued) function valued in $\mathrm{Hom}(V_{\lambda_2}\otimes V_{\lambda_1}, \mathrm{Hom}(\mathscr{H}_{\mu_s}, \hat{\mathscr{H}}_{\mu_t}))$ on the complex manifold $M_2 = \{(z_2,z_1)\in (\mathbb{C}\setminus\{0\})^2;\ z_1\neq z_2\}$.

Denote by $\Phi^2(u_2;z_1)\Phi^1(u_1,z_2) = C_\gamma\ (\Phi^2(u_2;z_2)\Phi^1(u_1,z_1))$ its analytic continuation along the path γ:

$$\gamma(t) = \left(\frac{z_2+z_1}{2} + e^{\pi\sqrt{-1}t}\,\frac{z_2-z_1}{2},\ \frac{z_2+z_1}{2} - e^{\pi\sqrt{-1}t}\,\frac{z_2-z_1}{2}\right),\quad t\in[0,1]$$

for $(w,z)\in\mathscr{R}$, then the corresponding analytic continuation

$$T\ \langle \Phi_\nu(z_t)\Phi^2(z_1)\Phi^1(z_2)\Phi_{id}(z_s)\rangle$$

satisfies the systems $KZ(T\Lambda)$ and $\ell C(T\Lambda)$ as a $\mathrm{Hom}(V_{\lambda_1}\otimes V_{\lambda_2}, \mathrm{Hom}(\mathscr{H}_{\mu_s}, \hat{\mathscr{H}}_{\mu_t}))$-valued function, where T is the transposition operator: $\mathrm{Hom}(V_{\lambda_2}\otimes V_{\lambda_1}, A) \longrightarrow \mathrm{Hom}(V_{\lambda_1}\otimes V_{\lambda_2}, A)$,

$$(T\varphi)(u_1\otimes u_2) = \varphi(u_2\otimes u_1) \qquad (u_2\otimes u_1 \in V_{\lambda_1}\otimes V_{\lambda_2}),$$

and $T\Lambda = (\mu_t^\dagger, \lambda_1, \lambda_2, \mu_s)$. Hence the analytic continuation along γ gives an isomorphism between the spaces of compositions of vertex operators:

Theorem 3. (Commutation Relations)

For $\Lambda = (\mu_t^+, \lambda_2, \lambda_1, \mu_s)$, $C(\Lambda) = C_\gamma(\Lambda)$ is an isomorphism :

$$\begin{array}{ccc} C_\gamma(\Lambda):\ \mathscr{V}(\Lambda) & \longrightarrow & \mathscr{V}(T\Lambda) \\ \| \wr & & \| \wr \\ \sum\limits_{\mu\in P_\ell} \mathscr{V}\left(\begin{bmatrix} & \lambda_2 & \\ \mu_t & & \mu \end{bmatrix}\right)\otimes\mathscr{V}\left(\begin{bmatrix} & \lambda_1 & \\ \mu & & \mu_s \end{bmatrix}\right) & \longrightarrow & \sum\limits_{\mu\in P_\ell} \mathscr{V}\left(\begin{bmatrix} & \lambda_1 & \\ \mu_t & & \mu \end{bmatrix}\right)\otimes\mathscr{V}\left(\begin{bmatrix} & \lambda_2 & \\ \mu & & \mu_s \end{bmatrix}\right) \end{array} .$$

Remark. The isomorphisms $C_\gamma(\Lambda)$, $\Lambda\in P_\ell^4$ enjoy the braid relations: For any $N\geq 1$, $\mu_t, \mu_s \in P_\ell$, introduce the space

$$\mathscr{V}(N;\mu_t,\mu_s) = \sum_{\lambda_1,\cdots,\lambda_N\in P_\ell} \mathscr{V}(\mu_t^+,\lambda_N,\cdots,\lambda_1,\mu_s) .$$

Define the operators C_i $(1\leq i\leq N-1)$ on $\mathscr{V}(N;\mu_t,\mu_s)$ such that

$$C_i\ \mathscr{V}(\mu_t^+,\lambda_N,\cdots,\lambda_1,\mu_s) \subset \mathscr{V}(\mu_t^+,\lambda_N,\cdots,\lambda_i,\lambda_{i+1},\cdots,\lambda_1,\mu_s)$$

and

$$\begin{aligned} &C_i(\varphi_N\otimes\cdots\otimes\varphi_1) \\ &= \varphi_N\otimes\cdots\otimes\varphi_{i+2}\otimes C(\mu_{i+1},\lambda_{i+1},\lambda_i,\mu_{i-1})(\varphi_{i+1}\otimes\varphi_i)\otimes\varphi_{i-1}\otimes\cdots\otimes\varphi_1 \end{aligned}$$

for $\varphi_N\otimes\cdots\otimes\varphi_1 \in \mathscr{V}(\mu_t^+,\lambda_N,\cdots,\lambda_1,\mu_s)_{(\mu_{N-1},\cdots,\mu_1)}$

$$= \mathscr{V}\left(\begin{bmatrix} & \lambda_N & \\ \mu_t & & \mu_{N-1}\end{bmatrix}\right)\otimes\cdots\otimes\mathscr{V}\left(\begin{bmatrix} & \lambda_i & \\ \mu_i & & \mu_{i-1}\end{bmatrix}\right)\otimes\cdots\otimes\mathscr{V}\left(\begin{bmatrix} & \lambda_1 & \\ \mu_1 & & \mu_s\end{bmatrix}\right).$$

Then

$$C_i\ C_{i+1}C_i = C_{i+1}\ C_i\ C_{i+1}$$

as isomorphisms of $\mathscr{V}(N;\mu_t,\mu_s)$ to itself.

§4. The composition $\Phi^2(u_2;w)\Phi^1(u_1;z)$ is singular at $w=z$ and its behaviour near $w=z$ is described as follows.

For $\Lambda = (\mu_t^+,\lambda_2,\lambda_1,\mu_s)$, the space $V_g^\vee(\Lambda)$ has another decomposition

$$V_g^\vee(\Lambda) \xleftarrow[F]{\cong} \sum_{\nu\in P_+} \mathrm{Hom}_g(V_{\lambda_2}\otimes V_{\lambda_1},V_\nu)\otimes\mathrm{Hom}_g(V_\nu\otimes V_{\mu_s},V_{\mu_t}) ,$$

where the identification F is given by

$$F(\varphi_2\otimes\varphi_1)(u_2\otimes u_1\otimes u_s) = \varphi_1(\varphi_2(u_2\otimes u_1)\otimes u_s) \qquad (u_i\in V_{\lambda_i},u_s\in V_\mu),$$

For $\varphi_2\in\mathscr{V}\begin{bmatrix} \nu \\ \mu_t\mu_s\end{bmatrix}$ and $\varphi_1\in\mathscr{V}\begin{bmatrix} & \lambda_2 & \\ \nu & & \lambda_1\end{bmatrix}$, a "vertex operator" $\Phi^f_{\varphi_2\otimes\varphi_1}(z)$ of $\mathscr{H}_{\mu_s}$ to $\hat{\mathscr{H}}_{\mu_t}$ parametrized by $V_{\lambda_2}\otimes V_{\lambda_1}$ defined by

$$\Phi^f_{\varphi_2\otimes\varphi_1}(u_2\otimes u_1;z) = \Phi_{\varphi_1}(\varphi_2(u_2\otimes u_1);z) \qquad (u_i\in V_{\lambda_i}).$$

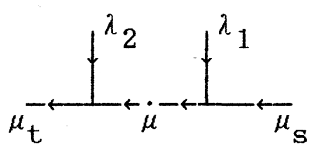
Theorem 4.(Short range expansion or Fusion rule)

i) Near $w=z$ $((w,z)\in\mathcal{R}_2)$,

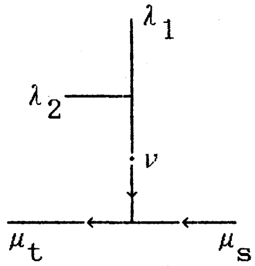
$$\Phi^2(u_2;w)\Phi^1(u_1;z) = \sum_{\nu\in P_\ell}(w-z)^{-\hat{\Delta}(w_1)}\left[\Phi^f_{\psi_\nu}(u_2\otimes u_1;z) + O(w-z)\right]$$

$$\sim (w-z)^{-(\Delta_{\lambda_1}+\Delta_{\lambda_2})}\sum_{\nu\in P_\ell}(w-z)^{\Delta_\nu}\,\Phi^f_{\psi_\nu}(u_2\otimes u_1;z)\ ,$$

where $\psi_\nu\in\mathcal{V}\left(\begin{bmatrix} & \lambda_2 & \\ \nu & & \lambda_1\end{bmatrix}\right)\otimes\mathcal{V}\left(\begin{bmatrix} & \nu & \\ \mu_t & & \mu_s\end{bmatrix}\right)$, *and* $O(w-z)$ *is holomorhic near* $w=z$ *an vanishes at* $w=z$:



$$\sim \sum_{\nu\in P_\ell}(w-z)^{-\hat{\Delta}(w_1(\nu))}$$



The value of $(w-z)^{-\hat{\Delta}(w_1)}$ *is chosen as it is positive for* $(w,z)\in\mathcal{R}_2\cap\mathbb{R}^2$.

ii) For $\Lambda = (\mu_t^+,\lambda_2,\lambda_1,\mu_s)$, *the fusion gives an isomorphism*

$$F(\Lambda):\mathcal{V}(\Lambda)\cong\sum_{\mu\in P_\ell}\mathcal{V}\left(\begin{bmatrix} & \lambda_2 & \\ \mu_t & & \mu\end{bmatrix}\right)\otimes\mathcal{V}\left(\begin{bmatrix} & \lambda_1 & \\ \mu & & \mu_s\end{bmatrix}\right)\longrightarrow\sum_{\nu\in P_\ell}\mathcal{V}\left(\begin{bmatrix} & \lambda_2 & \\ \nu & & \lambda_1\end{bmatrix}\right)\otimes\mathcal{V}\left(\begin{bmatrix} & \nu & \\ \mu_t & & \mu_s\end{bmatrix}\right)$$

defined by

$$F(\Lambda)(\varphi^2\otimes\varphi^1) = \sum_{\nu\in P_\ell}\psi_\nu \qquad (\varphi^2\otimes\varphi^1\in\mathcal{V}\left(\begin{bmatrix} & \lambda_2 & \\ \mu_t & & \mu\end{bmatrix}\right)\otimes\mathcal{V}\left(\begin{bmatrix} & \lambda_1 & \\ \mu & & \mu_s\end{bmatrix}\right)),$$

where ψ_ν *are the ones obtained in i) for* $\Phi^i = \Phi_\varphi{}^i$.

Theorem 5.

For $\Lambda = (\mu_t^*, \lambda_2, \lambda_1, \mu_s)$, *the following diagram commutes*:

$$\begin{tikzcd}
\mathcal{V}(\Lambda) \arrow[r, "F(\Lambda)"] \arrow[d, "C_\gamma"] & \sum_{\nu\in P_\ell} \mathcal{V}\left(\begin{bmatrix} \nu \\ \mu_t\ \mu_s \end{bmatrix}\right)\otimes\mathcal{V}\left(\begin{bmatrix} \lambda_2 \\ \nu\ \lambda_1 \end{bmatrix}\right) \arrow[d, "\mathrm{id}\otimes C_{\gamma_0}"] \\
\mathcal{V}(T\Lambda) \arrow[r, "F(T\Lambda)"] & \sum_{\nu\in P_\ell} \mathcal{V}\left(\begin{bmatrix} \nu \\ \mu_t\ \mu_s \end{bmatrix}\right)\otimes\mathcal{V}\left(\begin{bmatrix} \lambda_1 \\ \nu\ \lambda_2 \end{bmatrix}\right)
\end{tikzcd}$$

Remark. The equation $KZ(\Lambda)$ in the limit $z_4 \nearrow \infty$, $z_1 \searrow 0$ is reduced to a differential equation (reduced KZ-system) $RKZ(\Lambda)$ on $V_g(\Lambda)$- functions of one variable $\zeta = z_3/z_2$. The equation $RKZ(\Lambda)$ has only regular singularities at $\zeta=0,1,\infty$. The isomorphisms $C_\gamma(\Lambda)$ and $F(\Lambda)$ are essentially nothing but the connection matrices from the space of its solutions regularized at $\zeta=0$ to the spaces of solutions regularized at $\zeta=\infty$ and $\zeta=1$ respectively.

§5. Naturally arises a problem to determine the isomorphisms $C_\gamma(\Lambda)$ and $F(\Lambda)$, but it is very difficult to carry out for all cases. We succeeded (last year) in the case where $\hat{\mathfrak{g}}$ is an affine Lie algebra of type $A_n^{(1)}$ and $\Lambda = (\mu_t, \square, \square, \mu_s)$, where $\square$ means a Young diagram consisting of one node and represent the vector representation of $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbb{C})$.

Now let $\hat{\mathfrak{g}}$ be an affine Lie algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and P_+^0 be the set of weights $\lambda \in P_+$ such that the simple $\mathfrak{g}$-module V_λ can appear in some tensor products of the vector representations $V_\square$ of $\mathfrak{g}=\mathfrak{sl}(n+1;\mathbb{C})$, $\mathfrak{o}(2n;\mathbb{C})$, $\mathfrak{sp}(2n;\mathbb{C})$, $\mathfrak{o}(2n+1;\mathbb{C})$ respectively.

For each $\tau \in P_\ell$, introduce the space

$$\mathcal{V}_N(\tau) = \sum_{\mu} \mathcal{V}_N(\tau)_\mu ,$$

$$\mathcal{V}_N(\tau)_\mu = \mathcal{V}\left(\begin{bmatrix} & \square & \\ \tau & & \mu_{N-1}\end{bmatrix}\right)\otimes\cdots\otimes\mathcal{V}\left(\begin{bmatrix} & \square & \\ \mu_i & & \mu_{i-1}\end{bmatrix}\right)\otimes\cdots\otimes\mathcal{V}\left(\begin{bmatrix} & \square & \\ \mu_1 & & 0\end{bmatrix}\right),$$

where the summation is taken over the set $P_\ell^{N-1} \ni \mu = (\mu_1, \cdots, \mu_{N-1})$. Then $\mathcal{V}_N(\tau)$ is the subspace of $\mathcal{V}(N;\tau,0)$ which is invariant under the operators $C_i (1 \le i \le N-1)$.

The braid group B_N with N-strings of $\mathbb{C}$ has a system $\{b_i ; 1 \le i \le N-1\}$ of generators with the fundamental relations:

$$\text{(BR)} \quad \begin{cases} b_i b_j = b_j b_i & (|i-j| \ge 2) \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & (1 \le i \le N-2). \end{cases}$$

These generators b_i are represented by the curves on $\mathbb{C}$ defined by

$$b_i(t)=\left[N, N-1, \cdots, i+\tfrac{1}{2}(1+e^{\pi\sqrt{-1}t}), i+\tfrac{1}{2}(1-e^{\pi\sqrt{-1}t}), \cdots 2, 1\right]$$

$$t\in[0,1],$$

We now define a monodromy representation π_N^τ of B_N on the space $\mathcal{V}_N(\tau)$ as $\pi_N^\tau(b_i) = C_i$ $(1\le i\le N-1)$. The we get the main theorems.

Theorem 6.

If $\mathfrak{g}$ is of type A_n, then the monodromy representation $r^{1/(n+1)}\pi_N^\lambda$ in $\mathcal{V}_N(\tau)$ factors through the Iwahori-Hecke algebra $H_N(r)$, where $r=\exp(\frac{\pi\sqrt{-1}}{\ell+n+1})$.

Note. The algebra $H_N(r)$ is defined by generators $\{\tau_i, \tau_i^{-1}(1\le i\le N-1)\}$ with the defining relations:

$$\tau_i\tau_i^{-1}=\tau_i^{-1}\tau_i=1 \;,\quad \tau_i - \tau_i^{-1} = (r - r^{-1}) \quad (1\le i\le N-1)\;,$$
$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} \quad \text{and} \quad \tau_i\tau_j = \tau_j\tau_i \quad (|i-j|\ge 2).$$

Theorem 7. If $\mathfrak{g}$ is the simple Lie algebra of type B_n, C_n or D_n. Then the monodromy representation π_N^λ in $\mathcal{V}_N(\tau)$ factors through the Birmann-Wenzl-Murakami algebra $C_N(\mathfrak{g};r)$

where $r=\exp(\frac{\pi\sqrt{-1}}{\ell+g})$, $g(B_n) = 2n-1$, $g(C_n) = n+1$, $g(D_n) = 2n-2$; $C_N(B_n;r) = C_N(r^{-n-1/2},r)$, $C_N(C_n;r) = C_N(r^n,r)$ and $C_N(D_n;r)= C_N(r^{-n},r)$.

Note. The algebra $C_N(a,r)$ is defined by generators $\{\tau_i,\tau_i^{-1},\varepsilon_i\,(1\le i\le N-1)\}$ with the defining relations:

$$\tau_i\tau_i^{-1}=\tau_i^{-1}\tau_i=1\ ,\quad \tau_i\varepsilon_i = \varepsilon_i\tau_i = -(a^2r)^{-1}\varepsilon_i\ ,$$

$$\tau_i-\tau_i^{-1} = (r-r^{-1})(1-\varepsilon_i) \qquad (1\le i\le N-1),$$

$$\tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1},\ \varepsilon_i\varepsilon_{i+1}\varepsilon_i=\varepsilon_i,\ \varepsilon_{i+1}\varepsilon_i\varepsilon_{i+1} = \varepsilon_{i+1},$$

$$\tau_i^{\pm1}\varepsilon_{i+1}\varepsilon_i = \tau_{i+1}^{\mp1}\varepsilon_i\ ,\ \tau_{i+1}^{\pm1}\varepsilon_i\varepsilon_{i+1} = \tau_i^{\mp1}\varepsilon_{i+1}\ ,$$

$$\varepsilon_i\varepsilon_{i+1}\tau_i^{\pm1} = \varepsilon_i\tau_{i+1}^{\mp1}\ ,\ \varepsilon_{i+1}\varepsilon_i\tau_{i+1}^{\pm1} = \varepsilon_{i+1}\tau_i^{\mp1} \qquad (1\le i\le N-2),$$

$$\tau_i\tau_j = \tau_j\tau_i,\ \varepsilon_i\tau_j = \tau_j\varepsilon_i,\ \varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i \qquad (|i-j|\ge2),$$

The proof is carried out by the explicit calculation of a differential equation of 4-point function in a very special case and the algebraic arguments for the algebras $H_N(r)$ and $C_N(a,r)$.

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